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OSTROWSKI AND JENSEN TYPE INEQUALITIES FOR HIGHER DERIVATIVES WITH APPLICATIONS

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ABSTRACT. We consider inequalities which incorporate both Jensen and Ostrowski type inequalities for functions with absolutely continuous n -th derivatives. We provide applications of these inequalities for divergence measures. In particular, we obtain inequalities involving higher order χ -divergence.

1. INTRODUCTION

Jensen's inequality has been widely applied in many areas of research, e.g. probability theory, statistical physics, and information theory. The inequality was proved by Jensen in 1906 [13]: For a convex function $f : I \rightarrow \mathbb{R}$, the following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}, \quad a, b \in I. \quad (1.1)$$

Jensen's integral inequality takes the following form: for a μ -integrable function $g : \Omega \rightarrow [m, M] \subset \mathbb{R}$, and a convex function $f : [m, M] \rightarrow \mathbb{R}$, we have

$$f\left(\int_{\Omega} g d\mu\right) \leq \int_{\Omega} f \circ g d\mu. \quad (1.2)$$

Here, $(\Omega, \mathcal{A}, \mu)$ is a measurable space with $\int_{\Omega} d\mu = 1$, consisting of a set Ω , a σ -algebra \mathcal{A} of subsets of Ω , and a countably additive and positive measure μ on \mathcal{A} with values in the set of extended real numbers.

In 1938, Ostrowski proved the following inequality [12]:

Proposition 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_{\infty} := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_{\infty} (b-a), \quad (1.3)$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

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Dragomir [6] introduced some inequalities which combine the two aforementioned inequalities, referred to as the Jensen-Ostrowski type inequalities. We recall one of the results in the next proposition.

Proposition 1.2. *Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \in \overset{\circ}{I}$, the interior of I . If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and $\Phi \circ g, g \in L(\Omega, \mu)$, then*

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g \, d\mu - \Phi(x) - \lambda \left(\int_{\Omega} g \, d\mu - x \right) \right| \\ & \leq \int_{\Omega} |g - x| \|\Phi'((1 - \ell)x + \ell g - \lambda)\|_{[0,1],1} \, d\mu \\ & \leq \begin{cases} \|g - x\|_{\Omega, \infty} \|\Phi'((1 - \ell)x + \ell g - \lambda)\|_{[0,1],1} \|1\|_{\Omega,1}; \\ \|g - x\|_{\Omega,p} \|\Phi'((1 - \ell)x + \ell g - \lambda)\|_{[0,1],1} \|1\|_{\Omega,q}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|g - x\|_{\Omega,1} \|\Phi'((1 - \ell)x + \ell g - \lambda)\|_{[0,1],1} \|1\|_{\Omega, \infty}; \end{cases} \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

Here, ℓ denotes the identity function on $[0, 1]$, namely $\ell(t) = t$, for $t \in [0, 1]$. We also use the notation

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int_{\Omega} |k(t)|^p \, d\mu(t) \right)^{1/p}, & p \geq 1, \, k \in L_p(\Omega, \mu); \\ \operatorname{ess\,sup}_{t \in \Omega} |k(t)|, & p = \infty, \, k \in L_{\infty}(\Omega, \mu); \end{cases}$$

and

$$\|f\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |f(s)|^p \, ds \right)^{1/p}, & p \geq 1, \, f \in L_p([0, 1]); \\ \operatorname{ess\,sup}_{s \in [0,1]} |f(s)|, & p = \infty, \, f \in L_{\infty}([0, 1]). \end{cases}$$

Inequalities of Jensen and Ostrowski type are obtained by setting $x = \int_{\Omega} g \, d\mu$ and $\lambda = 0$, respectively, in Proposition 1.2. Further results on inequalities for functions with bounded derivatives and applications for f -divergence measures in information theory are also given in [6]. Similar inequalities are given for: (i) functions with derivatives that are of bounded variation and Lipschitz continuous in [7]; and (ii) functions which absolute values of the derivatives are convex in [8].

New inequalities of Jensen-Ostrowski type are given in the papers [2] and [3]. We recall one of the results in the following proposition:

Proposition 1.3 (Cerone, Dragomir, Kikianty [3]). *Let $f : I \rightarrow \mathbb{C}$ be a differentiable function on $\overset{\circ}{I}$, $f' : [a, b] \subset \overset{\circ}{I} \rightarrow \mathbb{C}$ is absolutely continuous on $[a, b]$, and $\zeta \in [a, b]$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω such that $f \circ g, g, (g - \zeta)^2 \in$*

$L(\Omega, \mu)$, with $\int_{\Omega} d\mu = 1$, then for any $\lambda \in \mathbb{C}$,

$$\begin{aligned} & \left| \int_{\Omega} f \circ g d\mu - f(\zeta) - \left(\int_{\Omega} g d\mu - \zeta \right) f'(\zeta) - \frac{1}{2} \lambda \int_{\Omega} (g - \zeta)^2 d\mu \right| \\ & \leq \frac{1}{2} \int_{\Omega} (g - \zeta)^2 \|f''((1 - \ell)\zeta + \ell g) - \lambda\|_{[0,1],\infty} d\mu \\ & \leq \begin{cases} \frac{1}{2} \|g - \zeta\|_{\Omega,\infty}^2 \|f''((1 - \ell)\zeta + \ell g) - \lambda\|_{[0,1],\infty} \|_{\Omega,1}; \\ \frac{1}{2} \|(g - \zeta)^2\|_{\Omega,p} \|f''((1 - \ell)\zeta + \ell g) - \lambda\|_{[0,1],\infty} \|_{\Omega,q}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \|(g - \zeta)^2\|_{\Omega,1} \|f''((1 - \ell)\zeta + \ell g) - \lambda\|_{[0,1],\infty} \|_{\Omega,\infty}. \end{cases} \end{aligned}$$

In this paper, we generalise the results in [3] (including Proposition 1.3) for functions with absolutely continuous n -th derivative. We start with some identities in Section 2 to assist us in proving our main theorems. We obtain our main results in Section 3: inequalities with bounds involving the p -norms ($1 \leq p \leq \infty$), inequalities for functions with further assumptions of bounded $(n+1)$ -th derivatives, and inequalities for functions where the absolute value of the $(n+1)$ -th derivative satisfies some convexity conditions. The case of $n = 1$ recovers the results in [3]. Applications for f -divergence measure are provided in Section 4.

2. IDENTITIES

Throughout the paper, we denote \mathring{I} to be the interior of the set I .

Lemma 2.1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{C}$ (I is an interval of \mathbb{R}) be such that $f^{(n)}$ is absolutely continuous on I , and $\zeta \in \mathring{I}$. If $g : \Omega \rightarrow I$ is Lebesgue μ -measurable on Ω and $f \circ g$, $(g - \zeta)^k$, $f^{(n+1)}((1 - s)\zeta + sg) \in L(\Omega, \mu)$ for all $k \in \{1, \dots, n+1\}$ and $s \in [0, 1]$, then we have*

$$\begin{aligned} & \int_{\Omega} f \circ g d\mu - f(\zeta) \\ & - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} d\mu - \lambda \frac{1}{(n+1)!} \int_{\Omega} (g - \zeta)^{n+1} d\mu \\ & = \frac{1}{n!} \int_{\Omega} (g - \zeta)^{n+1} \left(\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sg) - \lambda \right] ds \right) d\mu \\ & = \frac{1}{n!} \int_0^1 (1-s)^n \left(\int_{\Omega} (g - \zeta)^{n+1} \left[f^{(n+1)}((1-s)\zeta + sg) - \lambda \right] d\mu \right) ds, \end{aligned} \tag{2.1}$$

for any $\lambda \in \mathbb{C}$.

Proof. For all $x, \zeta \in \mathring{I}$ we have the Taylor's formula with integral remainder

$$f(x) = f(\zeta) + \sum_{k=1}^n \frac{(x - \zeta)^k}{k!} f^{(k)}(\zeta) + \frac{1}{n!} \int_{\zeta}^x (x - t)^n f^{(n+1)}(t) dt. \tag{2.2}$$

If we make the change of variable $t = (1 - s)\zeta + sx$, then $dt = (x - \zeta) ds$, and

$$x - t = x - (1 - s)\zeta - sx = (1 - s)(x - \zeta),$$

and from (2.2) we get

$$\begin{aligned} f(x) &= f(\zeta) + \sum_{k=1}^n \frac{(x-\zeta)^k}{k!} f^{(k)}(\zeta) \\ &\quad + \frac{1}{n!} (x-\zeta)^{n+1} \int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx) ds. \end{aligned} \quad (2.3)$$

On the other hand,

$$\begin{aligned} &\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda \right] ds \\ &= \int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx) ds - \lambda \int_0^1 (1-s)^n ds \\ &= \int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx) ds - \lambda \frac{1}{n+1}, \end{aligned}$$

therefore

$$\begin{aligned} &\int_0^1 (1-s)^n f^{(n+1)}((1-s)\zeta + sx) ds \\ &= \int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda \right] ds + \lambda \frac{1}{n+1}, \end{aligned}$$

and by (2.3) we get

$$\begin{aligned} f(x) &= f(\zeta) + \sum_{k=1}^n \frac{(x-\zeta)^k}{k!} f^{(k)}(\zeta) \\ &\quad + \frac{1}{n!} (x-\zeta)^{n+1} \left[\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda \right] ds + \lambda \frac{1}{n+1} \right] \\ &= f(\zeta) + \sum_{k=1}^n \frac{(x-\zeta)^k}{k!} f^{(k)}(\zeta) + \lambda \frac{1}{(n+1)!} (x-\zeta)^{n+1} \\ &\quad + \frac{1}{n!} (x-\zeta)^{n+1} \int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sx) - \lambda \right] ds. \end{aligned} \quad (2.4)$$

If $g : \Omega \rightarrow I$ is Lebesgue μ -measurable on Ω then by (2.4) we have

$$\begin{aligned} f(g(u)) &= f(\zeta) + \sum_{k=1}^n \frac{(g(u)-\zeta)^k}{k!} f^{(k)}(\zeta) + \lambda \frac{1}{(n+1)!} (g(u)-\zeta)^{n+1} \\ &\quad + \frac{1}{n!} (g(u)-\zeta)^{n+1} \int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sg(u)) - \lambda \right] ds, \end{aligned} \quad (2.5)$$

for all $u \in \Omega$.

Since $f \circ g, (g - \zeta)^k$, and $f^{(n+1)}((1-s)\zeta + sg) \in L(\Omega, \mu)$ for $k \in \{1, \dots, n+1\}$, $s \in [0, 1]$, we get the following by taking the integral in (2.5) and since $\int_{\Omega} d\mu = 1$:

$$\begin{aligned} & \int_{\Omega} f \circ g d\mu - f(\zeta) \\ & - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} d\mu - \lambda \frac{1}{(n+1)!} \int_{\Omega} (g - \zeta)^{n+1} d\mu \\ & = \frac{1}{n!} \int_{\Omega} (g - \zeta)^{n+1} \left(\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sg) - \lambda \right] ds \right) d\mu \\ & = \frac{1}{n!} \int_0^1 (1-s)^n \left(\int_{\Omega} (g - \zeta)^{n+1} \left[f^{(n+1)}((1-s)\zeta + sg) - \lambda \right] d\mu \right) ds, \end{aligned} \quad (2.6)$$

for any $\lambda \in \mathbb{C}$. We use Fubini's theorem for the last equality. \square

Remark. When $n = 1$ we have

$$\begin{aligned} & \int_{\Omega} f \circ g d\mu - f(\zeta) - f'(\zeta) \int_{\Omega} (g - \zeta) d\mu - \frac{1}{2} \lambda \int_{\Omega} (g - \zeta)^2 d\mu \\ & = \int_{\Omega} (g - \zeta)^2 \left(\int_0^1 (1-s) [f''((1-s)\zeta + sg) - \lambda] ds \right) d\mu \\ & = \int_0^1 (1-s) \left(\int_{\Omega} (g - \zeta)^2 [f''((1-s)\zeta + sg) - \lambda] d\mu \right) ds, \end{aligned}$$

for any $\lambda \in \mathbb{C}$, which recover the identities obtained in [3, Lemma 1]. Consequently, the results in this paper recover the associated ones in [3] by setting $n = 1$.

Corollary 2.2. Under the assumptions of Lemma 2.1, we have

$$\begin{aligned} & \int_{\Omega} f \circ g d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} d\mu \\ & = \frac{1}{n!} \int_{\Omega} (g - \zeta)^{n+1} \left(\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sg) \right] ds \right) d\mu \\ & = \frac{1}{n!} \int_0^1 (1-s)^n \left(\int_{\Omega} (g - \zeta)^{n+1} \left[f^{(n+1)}((1-s)\zeta + sg) \right] d\mu \right) ds \end{aligned} \quad (2.7)$$

by setting $\lambda = 0$.

Remark. Another estimate one may obtain is to consider the mean value form of the remainder in (2.2)

$$f(x) = f(\zeta) + \sum_{k=1}^n \frac{(x - \zeta)^k}{k!} f^{(k)}(\zeta) + \frac{(x - \zeta)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad (2.8)$$

where ξ is between x and ζ . By setting $x = g(t)$ ($t \in \Omega$) and integrate (2.8) on Ω , we obtain

$$\begin{aligned} & \int_{\Omega} f \circ g d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} d\mu \\ & = \int_{\Omega} f^{(n+1)}(\xi) \frac{(g - \zeta)^{n+1}}{(n+1)!} d\mu \end{aligned} \quad (2.9)$$

where $\xi = \xi(t)$ is between $g(t)$ and ζ .

3. MAIN RESULTS

We denote by ℓ , the identity function on $[0, 1]$, namely, $\ell(t) = t$ ($t \in [0, 1]$). For $t \in \Omega$, $\zeta \in [a, b]$, and $\lambda \in \mathbb{C}$, we have

$$\operatorname{ess\,sup}_{s \in [0, 1]} |f^{(k)}((1-s)\zeta + sg(t)) - \lambda| = \|f^{(k)}((1-\ell)\zeta + \ell g) - \lambda\|_{[0, 1], \infty},$$

for all $k = 1, \dots, n+1$.

Theorem 3.1. *Let $f : I \in \mathbb{R} \rightarrow \mathbb{C}$ (I interval of \mathbb{R}) be such that $f^{(n)}$ is absolutely continuous on I and $\zeta \in \overset{\circ}{I}$. If $g : \Omega \rightarrow I$ is Lebesgue μ -measurable on Ω and $f \circ g$, $(g - \zeta)^k$, $f^{(n+1)}((1-s)\zeta + sg) \in L(\Omega, \mu)$ for all $k \in \{1, \dots, n+1\}$ and $s \in [0, 1]$, then we have*

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) \right. \\ & \quad \left. - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} \, d\mu - \lambda \frac{1}{(n+1)!} \int_{\Omega} (g - \zeta)^{n+1} \, d\mu \right| \\ & \leq \frac{1}{(n+1)!} \left(\int_{\Omega} |g - \zeta|^{n+1} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) - \lambda \right\|_{[0, 1], \infty} \, d\mu \right) \\ & \leq \begin{cases} \frac{1}{(n+1)!} \| |g - \zeta|^{n+1} \|_{\Omega, \infty} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) - \lambda \right\|_{[0, 1], \infty} \Big\|_{\Omega, 1}, \\ \frac{1}{(n+1)!} \| |g - \zeta|^{n+1} \|_{\Omega, p} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) - \lambda \right\|_{[0, 1], \infty} \Big\|_{\Omega, q}, \\ \frac{1}{(n+1)!} \| |g - \zeta|^{n+1} \|_{\Omega, 1} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) - \lambda \right\|_{[0, 1], \infty} \Big\|_{\Omega, \infty}, \end{cases} \\ & \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1 \end{aligned} \tag{3.1}$$

for any $\lambda \in \mathbb{C}$.

Proof. Taking the modulus in (2.6), we have

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) \right. \\ & \quad \left. - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} \, d\mu - \lambda \frac{1}{(n+1)!} \int_{\Omega} (g - \zeta)^{n+1} \, d\mu \right| \\ & \leq \frac{1}{n!} \int_0^1 (1-s)^n \left(\int_{\Omega} |g - \zeta|^{n+1} \left| f^{(n+1)}((1-s)\zeta + sg) - \lambda \right| \, d\mu \right) ds \\ & \leq \frac{1}{n!} \int_0^1 (1-s)^n \, ds \left(\int_{\Omega} |g - \zeta|^{n+1} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) - \lambda \right\|_{[0, 1], \infty} \, d\mu \right) \\ & \leq \frac{1}{(n+1)!} \left(\int_{\Omega} |g - \zeta|^{n+1} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) - \lambda \right\|_{[0, 1], \infty} \, d\mu \right), \end{aligned}$$

for any $\lambda \in \mathbb{C}$. We obtain the desired result by applying Hölder's inequality. \square

Corollary 3.2. *Under the assumptions of Theorem 3.1, we have the following Ostrowski type inequality:*

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\ & \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} \int_{\Omega} |g-\zeta|^{n+1} \, d\mu. \end{aligned} \quad (3.2)$$

We also have the following Jensen type inequality:

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f\left(\int_{\Omega} g \, d\mu\right) - \sum_{k=2}^n f^{(k)}\left(\int_{\Omega} g \, d\mu\right) \int_{\Omega} \frac{(g - \int_{\Omega} g \, d\mu)^k}{k!} \, d\mu \right| \\ & \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} \int_{\Omega} \left| g - \int_{\Omega} g \, d\mu \right|^{n+1} \, d\mu. \end{aligned} \quad (3.3)$$

Proof. We have from (3.1) with $\lambda = 0$

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\ & \leq \frac{1}{(n+1)!} \left(\int_{\Omega} |g-\zeta|^{n+1} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) \right\|_{[0,1],\infty} \, d\mu \right). \end{aligned}$$

For any $t \in \Omega$ and almost every $s \in [0, 1]$, we have

$$|f^{(n+1)}((1-s)\zeta + sg(t))| \leq \operatorname{ess\,sup}_{u \in I} |f^{(n+1)}(u)| = \|f^{(n+1)}\|_{I,\infty}.$$

Therefore, we have

$$\begin{aligned} \left\| f^{(n+1)}((1-\ell)\zeta + \ell g) \right\|_{[0,1],\infty} & \leq \operatorname{ess\,sup}_{s \in [0,1], \, t \in \Omega} \|f^{(n+1)}((1-s)\zeta + sg(t))\| \\ & \leq \|f^{(n+1)}\|_{I,\infty}. \end{aligned}$$

Thus,

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\ & \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} \int_{\Omega} |g-\zeta|^{n+1} \, d\mu. \end{aligned}$$

The proof is completed. \square

Alternative proof for Corollary 3.2. From (2.9), we have the following for $\xi = \xi(t)$ is between $g(t)$ and ζ , where $t \in \Omega$:

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\ & = \left| \int_{\Omega} f^{(n+1)}(\xi) \frac{(g-\zeta)^{n+1}}{(n+1)!} \, d\mu \right| \\ & \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I,\infty} \int_{\Omega} |g-\zeta|^{n+1} \, d\mu. \end{aligned}$$

This completes the proof. \square

Remark (Ostrowski type inequality). *Let $\Omega = [a, b]$, $g : [a, b] \rightarrow [a, b]$ defined by $g(t) = t$, and $\mu(t) = t/(b-a)$. We have*

$$\begin{aligned}
& \left| \int_{\Omega} f \circ g d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} d\mu \right| \\
&= \left| \frac{1}{b-a} \int_a^b f(t) dt - f(\zeta) - \frac{1}{b-a} \sum_{k=1}^n f^{(k)}(\zeta) \int_a^b \frac{(t-\zeta)^k}{k!} dt \right| \\
&= \left| \frac{1}{b-a} \int_a^b f(t) dt - f(\zeta) - \frac{1}{b-a} \frac{1}{(n+1)!} \sum_{k=1}^n f^{(k)}(\zeta) \left[(b-\zeta)^{k+1} - (a-\zeta)^{k+1} \right] \right| \\
&\leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{[a,b],\infty} \frac{1}{b-a} \int_a^b |t-\zeta|^{n+1} dt \\
&= \frac{1}{(n+2)!} \|f^{(n+1)}\|_{[a,b],\infty} \frac{[(\zeta-a)^{n+2} + (b-\zeta)^{n+2}]}{b-a}.
\end{aligned}$$

For the next result, we need the following notation and proposition: for $\gamma, \Gamma \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions [6]

$$\bar{U}_{[a,b]}(\gamma, \Gamma) := \left\{ h : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re} \left[(\Gamma - h(t))(\overline{h(t)} - \bar{\gamma}) \right] \geq 0 \text{ for a.e. } t \in [a, b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ h : [a, b] \rightarrow \mathbb{C} \mid \left| h(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a, b] \right\}.$$

The following representation results may be stated [6].

Proposition 3.3. *For any $\gamma, \Gamma \in \mathbb{C}$ and $\gamma \neq \Gamma$, we have*

- (i) $\bar{U}_{[a,b]}(\gamma, \Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ are nonempty, convex and closed sets;
- (ii) $\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$; and
- (iii) $\bar{U}_{[a,b]}(\gamma, \Gamma) = \left\{ h : [a, b] \rightarrow \mathbb{C} \mid (\operatorname{Re}(\Gamma) - \operatorname{Re}(h(t))) (\operatorname{Re}(h(t)) - \operatorname{Re}(\gamma)) \right. \\ \left. + (\operatorname{Im}(\Gamma) - \operatorname{Im}(h(t))) (\operatorname{Im}(h(t)) - \operatorname{Im}(\gamma)) \geq 0 \text{ for a.e. } t \in [a, b] \right\}.$

We have the following Jensen-Ostrowski inequality for functions with bounded higher $(n+1)$ -th derivatives:

Theorem 3.4. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{C}$ (I interval of \mathbb{R}) be such that $f^{(n)}$ is absolutely continuous on I and $\zeta \in \overset{\circ}{I}$. For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $f^{(n+1)} \in \bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$. If $g : \Omega \rightarrow I$ is Lebesgue μ -measurable on Ω and $f \circ g, (g-\zeta)^k, f^{(n+1)}((1-s)\zeta + sg) \in L(\Omega, \mu)$ for all $k \in \{1, \dots, n+1\}$ and $s \in [0, 1]$, then we have*

$$\begin{aligned}
& \left| \int_{\Omega} f \circ g d\mu - f(\zeta) \right. \\
& \quad \left. - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} d\mu - \frac{\gamma + \Gamma}{2} \frac{1}{(n+1)!} \int_{\Omega} (g-\zeta)^{n+1} d\mu \right| \\
& \leq \frac{1}{2(n+1)!} |\Gamma - \gamma| \int_{\Omega} |g - \zeta|^{n+1} d\mu.
\end{aligned} \tag{3.4}$$

Proof. Let $\lambda = (\gamma + \Gamma)/2$ in (2.6), we have

$$\begin{aligned} & \int_{\Omega} f \circ g \, d\mu - f(\zeta) \\ & - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} \, d\mu - \frac{\gamma + \Gamma}{2} \frac{1}{(n+1)!} \int_{\Omega} (g - \zeta)^{n+1} \, d\mu \\ & = \frac{1}{n!} \int_{\Omega} (g - \zeta)^{n+1} \left(\int_0^1 (1-s)^n \left[f^{(n+1)}((1-s)\zeta + sg) - \frac{\gamma + \Gamma}{2} \right] ds \right) d\mu \end{aligned}$$

Since $f^{(n+1)} \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, we have

$$\left| f^{(n+1)}((1-s)\zeta + sg) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|, \quad (3.5)$$

for almost every $s \in [0, 1]$ and $t \in \Omega$. Multiply (3.5) with $(1-s)^n > 0$ and integrate over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg) - \frac{\gamma + \Gamma}{2} \right| ds \\ & \leq \frac{1}{2} |\Gamma - \gamma| \int_0^1 (1-s)^n \, ds = \frac{1}{2(n+1)} |\Gamma - \gamma|, \end{aligned}$$

for any $t \in \Omega$. Now, we have

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) \right. \\ & \left. - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} \, d\mu - \frac{\gamma + \Gamma}{2} \frac{1}{(n+1)!} \int_{\Omega} (g - \zeta)^{n+1} \, d\mu \right| \\ & \leq \frac{1}{2(n+1)!} |\Gamma - \gamma| \int_{\Omega} |g - \zeta|^{n+1} \, d\mu. \end{aligned}$$

This completes the proof. \square

Corollary 3.5. *When $\zeta = (a + b)/2$ in Theorem 3.4, we have the following Ostrowski inequality:*

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f\left(\frac{a+b}{2}\right) - \sum_{k=1}^n f^{(k)}\left(\frac{a+b}{2}\right) \int_{\Omega} \frac{(g - \frac{a+b}{2})^k}{k!} \, d\mu \right. \\ & \left. - \frac{\gamma + \Gamma}{2} \frac{1}{(n+1)!} \int_{\Omega} \left(g - \frac{a+b}{2}\right)^{n+1} \, d\mu \right| \\ & \leq \frac{1}{2(n+1)!} |\Gamma - \gamma| \int_{\Omega} \left| g - \frac{a+b}{2} \right|^{n+1} \, d\mu. \end{aligned}$$

When $\zeta = \int_{\Omega} g d\mu$ in Theorem 3.4, we have the following Jensen type inequality:

$$\begin{aligned} & \left| \int_{\Omega} f \circ g d\mu - f \left(\int_{\Omega} g d\mu \right) \right. \\ & \quad \left. - \sum_{k=1}^n f^{(k)} \left(\int_{\Omega} g d\mu \right) \int_{\Omega} \frac{(g - \zeta)^k}{k!} d\mu - \frac{\gamma + \Gamma}{2} \frac{1}{(n+1)!} \int_{\Omega} \left(g - \int_{\Omega} g d\mu \right)^{n+1} d\mu \right| \\ & \leq \frac{1}{2(n+1)!} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right|^{n+1} d\mu. \end{aligned}$$

We recall the following definition:

Definition. Let $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function. Then,

(1) h is convex, if for any $x, y \in I$ and $s \in [0, 1]$, we have

$$h((1-s)x + sy) \leq (1-s)h(x) + sh(y).$$

(2) h is quasi-convex, if for any $x, y \in I$ and $s \in [0, 1]$, we have

$$h((1-s)x + sy) \leq \max\{h(x), h(y)\}.$$

(3) h is log-convex, if for any $x, y \in I$ and $s \in [0, 1]$, we have

$$h((1-s)x + sy) \leq h(x)^{1-s} h(y)^s.$$

(4) for a fixed $q \in (0, 1]$, h is q -convex, if for any $x, y \in I$ and $s \in [0, 1]$, we have

$$h((1-s)x + sy) \leq (1-s)^q h(x) + s^q h(y).$$

We refer the reader to the paper by Dragomir [9], for further background on these notions of convexity.

We also need the following lemma to assist us in our calculations.

Lemma 3.6. For $\alpha, \beta \in \mathbb{R}$ and $n \geq 1$, we have

$$\int_0^1 (1-s)^n \left(\frac{\beta}{\alpha} \right)^s ds = -\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n! \frac{\frac{\beta}{\alpha} - 1}{(\log(\frac{\beta}{\alpha}))^{n+1}}. \quad (3.6)$$

Proof. For $n = 1$, integrating by parts gives us

$$\begin{aligned} \int_0^1 (1-s) \left(\frac{\beta}{\alpha} \right)^s ds &= \left. \frac{(1-s)}{\log(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^s \right|_0^1 + \frac{1}{\log(\frac{\beta}{\alpha})} \int_0^1 \left(\frac{\beta}{\alpha} \right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} + \frac{1}{(\log(\frac{\beta}{\alpha}))^2} \left(\frac{\beta}{\alpha} - 1 \right). \end{aligned}$$

For $n = 2$, integrating by parts gives us

$$\begin{aligned} \int_0^1 (1-s)^2 \left(\frac{\beta}{\alpha} \right)^s ds &= \left. \frac{(1-s)^2}{\log(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha} \right)^s \right|_0^1 + \frac{2}{\log(\frac{\beta}{\alpha})} \int_0^1 (1-s) \left(\frac{\beta}{\alpha} \right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \frac{2}{(\log(\frac{\beta}{\alpha}))^2} + \frac{2}{(\log(\frac{\beta}{\alpha}))^3} \left(\frac{\beta}{\alpha} - 1 \right). \end{aligned}$$

We assume that for n , we have

$$\begin{aligned} & \int_0^1 (1-s)^n \left(\frac{\beta}{\alpha}\right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n! \frac{\frac{\beta}{\alpha} - 1}{(\log(\frac{\beta}{\alpha}))^{n+1}}. \end{aligned}$$

We have

$$\begin{aligned} & \int_0^1 (1-s)^{n+1} \left(\frac{\beta}{\alpha}\right)^s ds \\ &= \frac{(1-s)^{n+1}}{\log(\frac{\beta}{\alpha})} \left(\frac{\beta}{\alpha}\right)^s \Big|_0^1 + \frac{n+1}{\log(\frac{\beta}{\alpha})} \int_0^1 (1-s)^n \left(\frac{\beta}{\alpha}\right)^s ds \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} + \frac{n+1}{\log(\frac{\beta}{\alpha})} \left[-\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n! \frac{\frac{\beta}{\alpha} - 1}{(\log(\frac{\beta}{\alpha}))^{n+1}} \right] \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \frac{n+1}{\log(\frac{\beta}{\alpha})^2} - \sum_{i=1}^{n-1} \frac{\frac{(n+1)!}{(n+1-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+2}} + (n+1)! \frac{\frac{\beta}{\alpha} - 1}{(\log(\frac{\beta}{\alpha}))^{n+2}} \\ &= -\frac{1}{\log(\frac{\beta}{\alpha})} - \sum_{i=1}^n \frac{\frac{(n+1)!}{(n+1-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + (n+1)! \frac{\frac{\beta}{\alpha} - 1}{(\log(\frac{\beta}{\alpha}))^{n+2}}, \end{aligned}$$

and this completes the proof. \square

In the next theorem, we assume that $|f^{(n+1)}|$ satisfies some convexity properties.

Theorem 3.7. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{C}$ (I interval of \mathbb{R}) be such that $f^{(n)}$ is absolutely continuous on I and $\zeta \in \overset{\circ}{I}$. Suppose that $g : \Omega \rightarrow I$ is Lebesgue μ -measurable on Ω and $f \circ g$, $(g - \zeta)^k$, $f^{(n+1)}((1-s)\zeta + sg) \in L(\Omega, \mu)$ for all $k \in \{1, \dots, n+1\}$.*

(i) *If $|f^{(n+1)}|$ is convex, then we have*

$$\begin{aligned} & \left| \int_{\Omega} f \circ g d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} d\mu \right| \\ & \leq \frac{1}{n!} \frac{1}{n+2} \left[|f^{(n+1)}(\zeta)| \int_{\Omega} |g - \zeta|^{n+1} d\mu + \frac{1}{(n+1)} \int_{\Omega} |g - \zeta|^{n+1} |f^{(n+1)} \circ g| d\mu \right]. \end{aligned}$$

(ii) *If $|f^{(n+1)}|$ is quasi-convex, then we have*

$$\begin{aligned} & \left| \int_{\Omega} f \circ g d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g - \zeta)^k}{k!} d\mu \right| \\ & \leq \frac{1}{(n+1)!} \max \left\{ |f^{(n+1)}(\zeta)| \int_{\Omega} |g - \zeta|^{n+1} d\mu, \int_{\Omega} |g - \zeta|^{n+1} |f^{(n+1)}(g(t))| d\mu \right\}. \end{aligned}$$

(iii) If $|f^{(n+1)}|$ is log-convex, then we have

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\ & \leq \frac{1}{n!} \int_{\Omega} |g-\zeta|^{n+1} \left[-\frac{|f^{(n+1)}(\zeta)|}{\log\left(\frac{|f^{(n+1)} \circ g|}{|f^{(n+1)}(\zeta)|}\right)} \right. \\ & \quad \left. - |f^{(n+1)}(\zeta)| \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log\left(\frac{|f^{(n+1)} \circ g|}{|f^{(n+1)}(\zeta)|}\right))^{i+1}} + n! \frac{|f^{(n+1)} \circ g| - |f^{(n+1)}(\zeta)|}{(\log\left(\frac{|f^{(n+1)} \circ g|}{|f^{(n+1)}(\zeta)|}\right))^{n+1}} \right] d\mu. \end{aligned}$$

(iv) If $|f^{(n+1)}|$ is q -convex (for a fixed $q \in (0, 1]$), then we have

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\ & \leq \frac{1}{n!} \frac{1}{n+q+1} \left[|f^{(n+1)}(\zeta)| \int_{\Omega} |g-\zeta|^{n+1} \, d\mu \right. \\ & \quad \left. + \frac{n}{(q+1)} \int_{\Omega} |g-\zeta|^{n+1} |f^{(n+1)} \circ g| \, d\mu \right]. \end{aligned}$$

Proof. (i) If $|f^{(n+1)}|$ is convex, then

$$\left| f^{(n+1)}((1-s)\zeta + sg(t)) \right| \leq (1-s)|f^{(n+1)}(\zeta)| + s|f^{(n+1)}(g(t))|,$$

for all $t \in \Omega$, which implies that

$$\begin{aligned} & \int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg(t)) \right| \, ds \\ & \leq \left[\int_0^1 (1-s)^{n+1} \, ds \right] |f^{(n+1)}(\zeta)| + \left[\int_0^1 s(1-s)^n \, ds \right] |f^{(n+1)}(g(t))| \\ & = \frac{1}{n+2} |f^{(n+1)}(\zeta)| + \frac{1}{(n+1)(n+2)} |f^{(n+1)}(g(t))|. \end{aligned}$$

Thus,

$$\begin{aligned} & \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\ & \leq \frac{1}{n!} \int_{\Omega} |g-\zeta|^{n+1} \left(\int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg) \right| \, ds \right) d\mu \\ & \leq \frac{1}{n!} \frac{1}{n+2} \left[|f^{(n+1)}(\zeta)| \int_{\Omega} |g-\zeta|^{n+1} \, d\mu + \frac{1}{(n+1)} \int_{\Omega} |g-\zeta|^{n+1} |f^{(n+1)} \circ g| \, d\mu \right]. \end{aligned}$$

(ii) If $|f^{(n+1)}|$ is quasi-convex, then

$$\left| f^{(n+1)}((1-s)\zeta + sg(t)) \right| \leq \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\},$$

for all $t \in \Omega$, which implies that

$$\begin{aligned} & \int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg(t)) \right| ds \\ & \leq \left[\int_0^1 (1-s)^n ds \right] \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\} \\ & = \frac{1}{n+1} \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\}. \end{aligned}$$

Thus,

$$\begin{aligned} & \left| \int_{\Omega} f \circ g d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} d\mu \right| \\ & \leq \frac{1}{n!} \int_{\Omega} |g-\zeta|^{n+1} \left(\int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg) \right| ds \right) d\mu \\ & \leq \frac{1}{(n+1)!} \int_{\Omega} |g-\zeta|^{n+1} \max\{|f^{(n+1)}(\zeta)|, |f^{(n+1)}(g(t))|\} d\mu \\ & = \frac{1}{(n+1)!} \max \left\{ |f^{(n+1)}(\zeta)| \int_{\Omega} |g-\zeta|^{n+1} d\mu, \int_{\Omega} |g-\zeta|^{n+1} |f^{(n+1)}(g(t))| d\mu \right\}. \end{aligned}$$

(iii) If $|f^{(n+1)}|$ is log-convex, then

$$\left| f^{(n+1)}((1-s)\zeta + sg(t)) \right| \leq |f^{(n+1)}(\zeta)|^{1-s} |f^{(n+1)}(g(t))|^s,$$

for all $t \in \Omega$, which implies that

$$\begin{aligned} & \int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg(t)) \right| ds \\ & \leq \left[\int_0^1 (1-s)^n |f^{(n+1)}(\zeta)|^{1-s} |f^{(n+1)}(g(t))|^s ds \right]. \end{aligned}$$

Let $\alpha := |f^{(n+1)}(\zeta)|$ and $\beta = \beta(t) := |f^{(n+1)}(g(t))|$. Since α does not depend on t , we have

$$\int_0^1 (1-s)^n \alpha^{1-s} \beta^s ds = \alpha \int_0^1 (1-s)^n \left(\frac{\beta}{\alpha} \right)^s ds.$$

By Lemma 3.6, we have

$$\begin{aligned} & \int_0^1 (1-s)^n \alpha^{1-s} \beta^s ds \\ & = \alpha \int_0^1 (1-s)^n \left(\frac{\beta}{\alpha} \right)^s ds \\ & = -\frac{\alpha}{\log(\frac{\beta}{\alpha})} - \alpha \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{\beta}{\alpha}))^{i+1}} + n! \frac{\beta - \alpha}{(\log(\frac{\beta}{\alpha}))^{n+1}}, \end{aligned}$$

and therefore

$$\begin{aligned}
& \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\
& \leq \frac{1}{n!} \int_{\Omega} |g-\zeta|^{n+1} \left(\int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg) \right| \, ds \right) \, d\mu \\
& \leq \frac{1}{n!} \int_{\Omega} |g-\zeta|^{n+1} \left[\int_0^1 (1-s)^n |f^{(n+1)}(\zeta)|^{1-s} |f^{(n+1)}(g(t))|^s \, ds \right] \, d\mu \\
& \leq \frac{1}{n!} \int_{\Omega} |g-\zeta|^{n+1} \left[-\frac{|f^{(n+1)}(\zeta)|}{\log\left(\frac{|f^{(n+1)} \circ g|}{|f^{(n+1)}(\zeta)|}\right)} \right. \\
& \quad \left. - |f^{(n+1)}(\zeta)| \sum_{i=1}^{n-1} \frac{\frac{n!}{(n-i)!}}{(\log(\frac{|f^{(n+1)} \circ g|}{|f^{(n+1)}(\zeta)|}))^{i+1}} + n! \frac{|f^{(n+1)} \circ g| - |f^{(n+1)}(\zeta)|}{(\log(\frac{|f^{(n+1)} \circ g|}{|f^{(n+1)}(\zeta)|}))^{n+1}} \right] \, d\mu.
\end{aligned}$$

(iv) If $|f^{(n+1)}|$ is q -convex (for a fixed $q \in (0, 1]$), then

$$|f^{(n+1)}((1-s)\zeta + sg)| \leq (1-s)^q |f^{(n+1)}(\zeta)| + s^q |f^{(n+1)}(g(t))|,$$

for all $t \in \Omega$, which implies that

$$\begin{aligned}
& \int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg(t)) \right| \, ds \\
& \leq \left[\int_0^1 (1-s)^{n+q} \, ds \right] |f^{(n+1)}(\zeta)| + \left[\int_0^1 (1-s)^n s^q \, ds \right] |f^{(n+1)}(g(t))| \\
& = \frac{1}{n+q+1} |f^{(n+1)}(\zeta)| + \frac{n}{(q+1)(n+q+1)} |f^{(n+1)}(g(t))|.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left| \int_{\Omega} f \circ g \, d\mu - f(\zeta) - \sum_{k=1}^n f^{(k)}(\zeta) \int_{\Omega} \frac{(g-\zeta)^k}{k!} \, d\mu \right| \\
& \leq \frac{1}{n!} \int_{\Omega} |g-\zeta|^{n+1} \left(\int_0^1 (1-s)^n \left| f^{(n+1)}((1-s)\zeta + sg) \right| \, ds \right) \, d\mu \\
& \leq \frac{1}{n!} \frac{1}{n+q+1} \left[|f^{(n+1)}(\zeta)| \int_{\Omega} |g-\zeta|^{n+1} \, d\mu \right. \\
& \quad \left. + \frac{n}{(q+1)} \int_{\Omega} |g-\zeta|^{n+1} |f^{(n+1)} \circ g| \, d\mu \right].
\end{aligned}$$

This completes the proof. \square

4. APPLICATIONS FOR f -DIVERGENCE

Assume that a set Ω and the σ -finite measure μ are given. Consider the set of all probability densities on μ to be

$$\mathcal{P} := \left\{ p|p : \Omega \rightarrow \mathbb{R}, p(t) \geq 0, \int_{\Omega} p(t) \, d\mu(t) = 1 \right\}.$$

We recall the definition of some divergence measures which we use in this text. The Kullback-Leibler divergence [10] is defined as:

$$D_{KL}(p, q) := \int_{\Omega} p(t) \log \left[\frac{p(t)}{q(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}. \quad (4.1)$$

Following is the definition of χ^2 -divergence:

$$D_{\chi^2}(p, q) := \int_{\Omega} p(t) \left[\left(\frac{q(t)}{p(t)} \right)^2 - 1 \right] d\mu(t), \quad p, q \in \mathcal{P}. \quad (4.2)$$

Following is the definition of the higher order χ -divergence [1]:

$$D_{\chi^k}(p, q) := \int_{\Omega} \frac{(q(t) - p(t))^k}{p^{k-1}(t)} d\mu(t), \quad p, q \in \mathcal{P}; \quad (4.3)$$

$$D_{|\chi|^k}(p, q) := \int_{\Omega} \frac{|q(t) - p(t)|^k}{p^{k-1}(t)} d\mu(t), \quad p, q \in \mathcal{P}. \quad (4.4)$$

The above definition(s) can be generalised as follows [11]:

$$D_{\chi^k, \lambda}(p, q) := \int_{\Omega} \frac{(q(t) - \lambda p(t))^k}{p^{k-1}(t)} d\mu(t), \quad p, q \in \mathcal{P}; \quad (4.5)$$

$$D_{|\chi|^k, \lambda}(p, q) := \int_{\Omega} \frac{|q(t) - \lambda p(t)|^k}{p^{k-1}(t)} d\mu(t), \quad p, q \in \mathcal{P}. \quad (4.6)$$

Csiszár f -divergence is defined as follows [4]

$$I_f(p, q) := \int_{\Omega} p(t) f \left[\frac{q(t)}{p(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}, \quad (4.7)$$

where f is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. The Kullback-Leibler divergence and the χ^2 -divergence are particular instances of Csiszár f -divergence. For the basic properties of Csiszár f -divergence, we refer the readers to [4], [5], and [14].

Proposition 4.1. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p, q \in \mathcal{P}$ and there exists constants $0 < r < 1 < R < \infty$ such that*

$$r \leq \frac{q(t)}{p(t)} \leq R, \quad \text{for } \mu\text{-a.e. } t \in \Omega. \quad (4.8)$$

If $\zeta \in [r, R]$ and $f^{(n)}$ is absolutely continuous on $[r, R]$, then we have the inequalities

$$\begin{aligned} & \left| I_f(p, q) - f(\zeta) - \sum_{k=1}^n \frac{1}{k!} f^{(k)}(\zeta) D_{\chi^k, \zeta}(p, q) \right| \\ & \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I, \infty} D_{|\chi|^{n+1}, \zeta}(p, q). \end{aligned}$$

In particular, when $\zeta = 1$, we have

$$\left| I_f(p, q) - \sum_{k=1}^n \frac{1}{k!} f^{(k)}(1) D_{\chi^k}(p, q) \right| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I, \infty} D_{|\chi|^{n+1}}(p, q). \quad (4.9)$$

We remark that we recover Theorem 1 of [1] in (4.9), with the assumption that $f(1) = 0$.

Proof. We choose $g(t) = q(t)/p(t)$ in (3.2), and note that $\int_{\Omega} p(t) d\mu = 1$. Therefore, we have

$$\begin{aligned}
& \left| \int_{\Omega} f\left(\frac{q(t)}{p(t)}\right) p(t) d\mu - f(\zeta) - \sum_{k=1}^n \frac{1}{k!} f^{(k)}(\zeta) \int_{\Omega} \left(\frac{q(t)}{p(t)} - \zeta\right)^k p(t) d\mu \right| \\
&= \left| I_f(p, q) - f(\zeta) - \sum_{k=1}^n \frac{1}{k!} f^{(k)}(\zeta) \int_{\Omega} \frac{(q(t) - \zeta p(t))^k}{p(t)^{k-1}} d\mu \right| \\
&= \left| I_f(p, q) - f(\zeta) - \sum_{k=1}^n \frac{1}{k!} f^{(k)}(\zeta) D_{\chi^k, \zeta}(p, q) \right| \\
&\leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I, \infty} \int_{\Omega} \left| \frac{q(t) - \zeta p(t)}{p(t)} \right|^{n+1} p(t) d\mu \\
&\leq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{I, \infty} D_{|\chi|^{n+1}, \zeta} d\mu.
\end{aligned}$$

This completes the proof. \square

Example. If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t \log(t)$, then

$$I_f(p, q) = \int_{\Omega} p(t) \frac{q(t)}{p(t)} \log\left(\frac{q(t)}{p(t)}\right) d\mu(t) = \int_{\Omega} q(t) \log\left(\frac{q(t)}{p(t)}\right) d\mu(t) = D_{KL}(q, p).$$

We have $f'(t) = \log(t) + 1$ and $f^{(k)}(t) = (-1)^k t^{-(k-1)}$, for $k \geq 2$. By Proposition 4.1, we have

$$\begin{aligned}
& \left| D_{KL}(q, p) - \zeta \log(\zeta) - (1 - \zeta)(\log(\zeta) + 1) - \sum_{k=2}^n \frac{1}{k!} (-1)^k \zeta^{-(k-1)} D_{\chi^k, \zeta}(p, q) \right| \\
&\leq \frac{1}{(n+1)!} r^{-n} D_{|\chi|^{n+1}, \zeta}(p, q),
\end{aligned}$$

for all $\zeta \in [r, R]$. When $\zeta = 1$, we have

$$\left| D_{KL}(q, p) - \sum_{k=2}^n \frac{1}{k!} (-1)^k D_{\chi^k}(p, q) \right| \leq \frac{1}{(n+1)!} r^{-n} D_{|\chi|^{n+1}}(p, q).$$

Example. If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = -\log(t)$, then

$$I_f(p, q) = - \int_{\Omega} p(t) \log\left(\frac{q(t)}{p(t)}\right) d\mu(t) = \int_{\Omega} p(t) \log\left(\frac{p(t)}{q(t)}\right) d\mu(t) = D_{KL}(p, q).$$

We have $f^{(k)}(t) = (-1)^k t^{-k}$ for $k \geq 1$. By Proposition 4.1, we have

$$\begin{aligned}
& \left| D_{KL}(p, q) + \log(\zeta) - \sum_{k=1}^n \frac{1}{k!} (-1)^k \zeta^{-k} D_{\chi^k, \zeta}(p, q) \right| \\
&\leq \frac{1}{(n+1)!} r^{-(n+1)} D_{|\chi|^{n+1}, \zeta}(p, q),
\end{aligned}$$

for all $\zeta \in [r, R]$. When $\zeta = 1$, we have

$$\left| D_{KL}(p, q) - \sum_{k=1}^n \frac{1}{k!} (-1)^k D_{\chi^k}(p, q) \right| \leq \frac{1}{(n+1)!} r^{-(n+1)} D_{|\chi|^{n+1}}(p, q).$$

REFERENCES

- [1] N. S. Barnett, P. Cerone and S. S. Dragomir, A. Sofo, *Approximating Csiszár f -divergence by the use of Taylor's formula with integral remainder*, Math. Inequal. Appl. **5** 3 (2002), 417–434.
- [2] P. Cerone, S. S. Dragomir and E. Kikianty, *Jensen-Ostrowski type inequalities and applications for f -divergence measures*, Appl. Math. Comput. **266** (2015), 304–315.
- [3] P. Cerone, S. S. Dragomir and E. Kikianty, *On inequalities of Jensen-Ostrowski type*, J. Inequal. Appl. **2015** (2015), Article 328.
- [4] I. I. Csiszár, *On topological properties of f -divergences*, Studia Math. Hungarica **2** (1967), 329–339.
- [5] I. I. Csiszár and J. Körner, *Information Theory: Coding Theorem for Discrete Memoryless Systems*, Academic Press (New York), (1981).
- [6] S. S. Dragomir, *Jensen and Ostrowski type inequalities for general Lebesgue integral with applications*, RGMIA Res. Rep. Coll. **17** (2014), Article 25.
- [7] S. S. Dragomir, *New Jensen and Ostrowski type inequalities for general Lebesgue integral with applications*, RGMIA Res. Rep. Coll. **7** (2014), Article 27.
- [8] S. S. Dragomir, *General Lebesgue integral inequalities of Jensen and Ostrowski type for differentiable functions whose derivatives in absolute value are h -convex and applications*, RGMIA Res. Rep. Coll. **17** (2014), Article 38.
- [9] S. S. Dragomir, *Integral inequalities of Jensen type for λ -convex functions*, RGMIA Res. Rep. Coll. **17** (2014), Article 18.
- [10] S. Kullback and R.A. Leibler, *On information and sufficiency*, Annals Math. Statist. **22** (1951), 79–86.
- [11] F. Nielsen and R. Nock, *On the Chi square and higher-order Chi distances for approximating f -divergences*, IEEE Signal Processing Letters **21** 1 (2014).
- [12] A. Ostrowski, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Hel. **10** (1938), 226–227.
- [13] J. E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Application*, Mathematics in Science and Engineering Vol. 187, Academic Press Inc. (Boston, MA), (1992).
- [14] I. Vajda, *Theory of Statistical Inference and Information*, Kluwer Academic Publishers, (Dordrecht-Boston), (1989).

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